ISSN 2229-5518

Morphological Space And Transform Systems

Ramkumar P.B, Pramod K.V

Abstract— Mathematical Morphology in its original form is a set theoretical approach to image analysis.It studies image transformations with a simple geometrical interpretation and their algebraic decomposition and synthesis in terms of elementary set operations.Mathematical Morphology has taken concepts and tools from different branches of Mathematics like algebra (lattice theory) ,Topology,Discrete geometry ,Integral Geometry,Geometrical Probability,Partial Differential Equations etc.In this paper ,a generalization of Morphological terms is introduced.In connection with algebraic generalization,Morphological operators can easily be defined by using this structure.This can provide information about operators and other tools within the system

Index Terms—morphological space, transform systems, slope transforms, legendre, kernel.

1 Generalized Structure for Mathematical Morphology

1.1 Definition: Morphogenetic field

LET X $\neq \varphi$ and W $\subseteq P(X)$ such that i) ϕ, X $\in$ W , ii) If B $\in$ W then its complement $\overline{B}$ $\in$ W iii) If $B_i \in$ W is a sequence of signals defined in X, then $\bigcup_{n=1}^{\infty} Bi \in$ W.

Let "A=$\{\phi : W \rightarrow U / \phi(\cup A_i) = \vee\phi(A_i) \,\&\, \phi(\wedge A_i) = \wedge\phi(A_i)\}$ " .Then W_U is called Morphogenetic field [14] where the family W_u is the set of all image signals defined on the continuous or discrete images Plane X and taking values in a set U .The pair (W_u, A) is called an operator space where A is the collection of operators defined on X.

1.2 Definition : Morphological Space

The triplet (X, W_u, A) consisting of a set X, a morphogenetic field W_u and an operator A(or collection of operators) defined on X is called a Morphological space.[14]

Example 1.If X = Z^2 then it is called Discrete Morphological space

Example 2.Let V be a complete lattice.

If X= V and A= (V, $\vee,\wedge.*,*'$) where $*,*'$ are dilation & erosion then (V, $\vee,\wedge.*,*'$) becomes a commutative complete lattice ordered double monoid or 'Clodum' [11] where (V, $*$) (V, $*'$) are commutative monoids.

Exampl 3.If (V, $*$) (V, $*'$) are groups then (V, $\vee,\wedge.*,*'$) is called a bounded lattice ordered group or blog.[11]

Proposition 1. *Clodum is an operator space.*

Since Clodum is a particular case as mentioned above , we can consider it as an operator space.

Proposition 2. *Blog is an operator space.*

Blog is another example for operator space. Similar particular cases exist corresponding to the algebra or geometry under consideration.

1.3 Definition:Dilation

Let$(L,\leq)$be a complete lattice, with infimum and minimum symbolized by$\wedge$ and$\vee$ respectively.[1]

A dilation is any operator $\delta: L \rightarrow L$ that distributes over the supremum and preserves the least element. $\vee_i \delta(X_i) = \delta(\vee_i X_i), \delta(\emptyset) = \emptyset$

1.4 Definition: Erosion

An erosion is any operator $\varepsilon : L \longrightarrow L$that distributes over the infimum [1]. $\wedge_i \varepsilon(X_i)=\varepsilon(\wedge_i X_i)$, $\varepsilon(U)$=U

1.5 Definition:Morphological Adjunctions

Let (X, W_u, A) & $(Y, W_u, \bar{A})$ be a morphological spaces.

The pair $(A, \bar{A})$ is called an adjunction iff $A(X) \leq Y \Leftrightarrow X \leq \bar{A}(Y)$ where $\bar{A}$ is an inverse operator of A.

Proposition 3.*Let* (X, W_u, δ) & (Y, W_u, ε) *be a morphological spaces with operators dilation and erosion on A. Then* $\delta(X) \leq Y \Leftrightarrow X \leq \varepsilon(Y)$.

Proposition 4(for lattice). *Let* (X, W_u, A) & $(Y, W_u, \bar{A})$ *be a morphological spaces. The pair* $(A, \bar{A})$ *is called an adjunction iff* $\forall u, v \in X, \exists$ *an adjunction* $(l_{u,v}, m_{v,u})$ *on U such that* $\bar{A}(x(u)) = \vee_{v \in X} m_{v,u}(x(v))$ *and* $A(y(v)) = \wedge_{u \in X} l_{u,v}(y(u))$, $\forall\ u, v \in X, x, y \in W_U$ *.[1]*

1.6 Definition:Morphological Kernel

The operator $\phi = \varepsilon \circ \delta$ defines a closure called morphological closure and $\phi^* = \delta \circ \varepsilon$ defines a kernel, called morphological kernel.

2 Slope transforms - generalization

2.1 Introduction

Fourier transforms are most useful linear signal transformations for quantifying the frequency content of signals and for analyzing their processing by linear time – invariant systems .They enable the analysis and design of linear time invariant systems (LTI)in the frequency domain.

Slope transforms are a special type of non linear signal transforms that can quantify the slope content of signals. It provide a transform domain for morphological systems. They are based on eigen functions of morphological systems that are lines parameterized by their slope. Dilation and Erosions are the fundamental operators in Mathematical Morphology. These operators are defined on lattice algebraic structure also. Based on this, Slope transforms are generally divided into three.

They are 1) A single valued slope transform for signals processed by erosion systems 2) A single valued slope transform for signals processed by dilation systems 3) A multi valued transform that results by replacing the suprema and infima of signals with the signal values at stationary points.

2.2 Special Case – Continuous Time Signals

All the three transforms stated above coincide for continuous-time signals that are convex or concave and have an invertible derivative. This become equal to the Legendre transform (irrespective of the difference due to the boundary conditions).

2.3 Morphological Signal Operators

The morphological signal operators are defined by using Lattice Dilation and Erosion of Signals. The morphological signal operators are parallel or serial inter connections of morphological dilation and erosions, respectively, defined as

$$(f + g)(x) = \vee_{y \in R^d} f(x - y) + g(y)$$
$$and (f + g)(x) = \wedge_{y \in R^d} f(x + y) - g(y) \quad (1)$$

Where $\vee$ denotes supremum and $\wedge$ denotes infimum.

2.4 Legendre Transforms

Let the signal $x(t)$ be concave and assume that there exist an invertible derivative $x' = \frac{dx}{dt}$. Imagine that the graph of x,not as a set of points $(t, x(t))$ but as the lower envelope of all its tangent lines . The Legendre transform [12] of x is based on this concept .The tangent at a point $(t, x(t))$ on the graph has slope and intercept equal to $X = x(t) - \alpha(t)$

$\therefore X_L(\alpha) = x[(x')^{-1}(\alpha)] - \alpha(x')^{-1}(\alpha)$ where f^{-1} denotes the inverse.

ISSN 2229-5518

The function X_L of the tangents intercept versus the slope is the Legendre transform of x [12]

and $x(t) = X_L[(X_L')^{-1}(-t)) + t(X_L')^{-1}(-t)]$ If the signal x is convex, then the signal is viewed as the upper envelope of its tangent lines.

2.5 Definition:Upper Slope Transform

For any signal $x : R \to \bar{R}$ its upper slope transform [12] is the function $X_{\vee} : R \to \bar{R}$ with $X_{\vee}(\alpha) = \vee_{t \in R} x(t) - \alpha t, \alpha \in R$ .The mapping between the signal and its transform is denoted by $A_{\vee} : x \to X_{\vee}$. If there is one to one correspondence between the signal and its transform, then it is denoted by $x(t) \xrightarrow{A_{\vee}} X_{\vee}(\alpha)$.

Proposition6. *Upper Slope Transform of a concave signal is equal to its Legendre Transform.*

Proof.Let $x(t)$ be a concave signal .Let it has an invertible derivative. For each real α , the intercept of the line passing from the point $(t, x(t))$ in the signals graph with slope α is given by $x(t) - \alpha t$.

For a fixed α , assume that t varies. Let there be a time instant t^* for which the intercept attains its maximum value. The intercept attains its maximum value when the line becomes tangent to the graph.Therefore $x'(t^*) = \alpha$.

Corresponding to the change in α , the tangent also changes,and the maximum intercept becomes a function of the slope α .By its definition ,the upper slope transform [12] is equal to this maximum intercept function. Thus, if the signal $x(t)$ is concave and has an invertible derivative,then the upper slope transform is equal to its Legendre transform.Hence the proof.

3 Results Based on the Generalized Structure

3.1 Definition:Self Conjugate Operator Space

An operator space (W_u, A) is called self conjugate if it has a negation.

Example4. A clodum V has conjugate a * for every 'a' such that (avb)* = a* $\wedge$ b* and (a$*$b)*' = (a* * ' b*) [11]

Example5. If V is a blog [4] then it becomes self conjugate by setting

$$a^* = \begin{cases} a^{-1}, \text{when} V \inf < a < V \sup \\ V \sup, \text{when} V \inf = a \\ V \inf, \text{when} V \sup = a \end{cases} \quad [11]$$

Example6. If X is a concave class then A* x (t)= x(-t) where A* = A $\wedge$ (A $\vee$) .

3.2 Definition:Self Conjugate Morphological Space

If the operator space (W_u, A) is self conjugate then the morphological space (X, W_u, A) is called a self conjugate morphological space.

3.3 Definition:Operatable Functions

Let (X, W_u, A) be a morphological space. The collection K (X, W, A) of operatable functions consists of all real valued morphologically operatable functions x(t) defined on X such that x(t) has finite operatability with respect to A.

A morphologically operatable function x$\in K$ iff $|x| \in K$ .ie. iff $|A(x(\alpha))| \leq A|x(\alpha)|$

3.4 Definition:Morphological Transform Systems

Let (X, W_u, A) be a perfect morphological space and K= K (X,W_u, A) be an operatable space. K is called a morphological transform system if $A[x_T(t)] = X(\alpha) \circ T(\alpha)$

Remark1.Since K is an operatable space,

1) $A[x(t) + y(t)] = X(\alpha) + Y(\alpha)$

2) $A[x_T(t)] = X(\alpha) \circ T(\alpha)$

3.5 Definition:Morphological Slope Transform System

If A= Av in the previous definition,then K is called a Morphological slope transform system where Av is the upper slope transform.

Let (X, W_u, A) be a self conjugate morphological space. If X is a concave class then A * (x(t))= x(-t) where A*= A $\wedge$ (A v) and A $\wedge$ is the lower slope transform. Also Av ($\vee x_c$)

= $\sum_{\forall c} A_\vee (x_c)$

Proposition 7(Characterization of Slope Transforms).

A Slope transform is an extended real valued function Av (or A $\wedge$) defined on a Morphogenetic field W_u such that

1. *Av (ϕ)= 0*
2. *Av (x_c) ≥ 0 $\forall x_c \in W_u$*
3. *Av is countably additive in the sense that if (x_c)is any disjoint sequence[or sampling Signal]then Av ($\vee x_c$)*

   = $\sum_{\forall c} A_\vee (x_c)$

Remark 2.Av takes $+\infty$ i.e Av (x_c) = ∞ if x(t) = ∞

Av (α) > $-\infty, \forall \alpha$ unless x(t) = $-\infty, \forall t$ t

If x=∞ - then Av = $-\infty$

Proposition 8. *Let K be a morphological transform system. Let X be a class of concave functions. Let $x(\alpha) \in X$ with each $x(\alpha)$ has an invertible derivative. Then $A_\vee(x(\alpha)) = L(x(\alpha))$ where L is the Legendre transform and Av is the upper slope transform.*

4 Conclusion

The slope transform has emerged as a transform which has similar properties with respect to morphological signal processing. Fourier transform does this with respect to linear signal processing .Main property of slope transform is that it transforms a supremal convolution (morphological dilation) into an addition .This is similar to the concept in Fourier transform transforms. In Fourier transform a linear convolution changed into a multiplication.

Difference between the Fourier transform and its morphological counterpart, the slope transform is that the Fourier transform is invertible but the slope transform only has an adjoint. In the sense of adjunctions, this means that the 'inverse' of the slope – transformed signal is not the original signal but only an approximation within the sub collection. In this paper we made an attempt for generalizing the algebraic structures related to the theory of Signal processing using Mathematical Morphology. Morphological operators can be redefined by using these structures. We hope that this will be helpful for finding new applications in Mathematical Morphology.

References

[1] John Goustias and Henk J.A.M Heijmans ,Mathematical Morphology, , I.O.S Press.

[2] H.J.A.M Heijmans, Morphological Image Operators,Boston, M.A Academic,1994 .

[3] J .Serra, Image Analysis and Mathematical Morphology,New York Academic ,1982.

[4] P .Maragos and R.W Schafer, "Morphological system for multi dimensional signal processing" Proc. IEEE,Vol,78,P.D 690-710,April 1990.

[5] P .Maragos,A representation theory for morphological image and signal processing. IEEE Transactions on Pattern analysis and machine intelligence 11,(1989),586-599.

[6] The Matheron Representation Theorem for Gray Scale Morphological Operators, G. CROMBEZ,Proceedings of the American Mathematical Society Volume 108, Number 3, March 1990(Proceedings)

[7] Heijmans, H.J.A.M ,and Ronse,C.The algebraic basis of Mathematical Morphology –Part I, Dilations and Erosions ,Computer vision, Graphics and Image Processing,50(1990) 245-295.

[8] Rein Van Den Boomgaard and Henk Heijmans, Morphological scale space- operators.

[9] Javier Vidal& Jos´e Crespo,Sets Matching in Binary Images Using Mathematical Morphology,International Conference of the Chilean Computer Science Society.

[10] Jean Cousty, Laurent Najman and Jean Serra,Some morphological operators in graph spaces,ISSM-2009.

[11] Petros Maragos,Lattice Image Processing: A Unification of Morphological and Fuzzy Algebraic Systems,Journal of Mathematical Imaging and Vision 22:333-353,2005.

ISSN 2229-5518

[12] Chu-Song Chen, Yi-Ping Hung,Theoretical Aspects of Vertically Invariant Gray-Level Morphological Operators and Their Application on Adaptive Signal and Image Filtering, IEEE transactions on signal processing, vol. 47, no. 4, april 1999 1049.
[13] Petros Maragos,Slope Transforms:Theory and Application to Nonlinear Signal Processing,IEEE Transactions on Signal Processing,Vol 43,No.4,April 1995.
[14] K.V Pramod, Ramkumar P.B , Convex Geometry and Mathematical Morphology, International Journal of Computer Applications,Vol:8,Page 40-45.

- *Ramkumar P.B is working as Assistant Professor in Mathematics at Rajagiri School of Engineering& Technology , Mahatma Gandhi University, India, PH-04842432058. E-mail: rkpbmaths@yahoo.co.in*
- *Pramod K.V is working as Professor at Department of Computer Applications,Cochin University, India, PH-01123456789. E-mail :pramodvijayaraghavan@gmail.com*